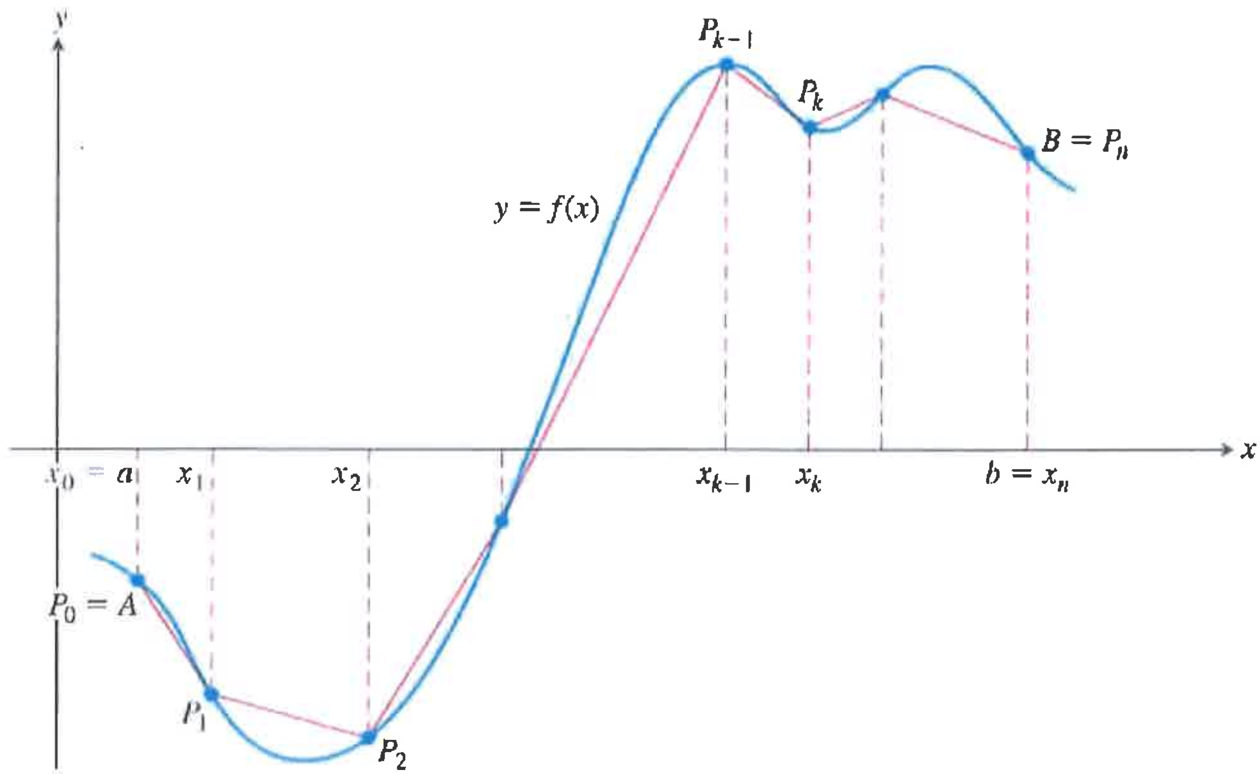
Arc Length



The distance from P_{k-1} to P_k

is $\sqrt{[x_k - x_{k-1}]^2 + [f(x_k) - f(x_{k-1})]^2}$

$$L \approx \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 \left(1 + \frac{[f(x_{k-1}) - f(x_{k-1})]^2}{[x_k - x_{k-1}]^2}\right)} =$$

$$\sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k$$

$$L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x)]^2} \; \Delta x_k$$

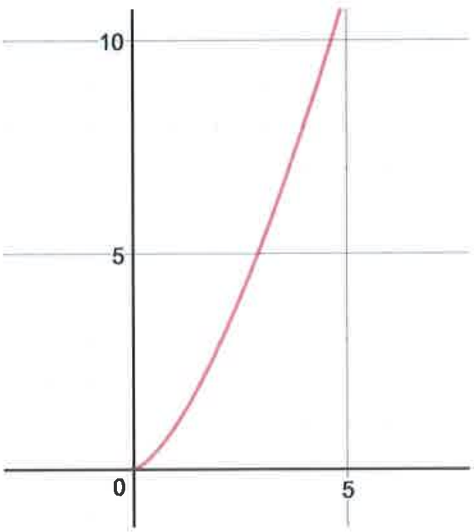
$$= \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

If f' is continuous on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx \,.$$

Example

Determine the length of the curve defined by $y = x^{3/2}$ over the interval $0 \leq x \leq 4$.



$y = f(x) = x^{\frac{3}{2}}$, $f'(x) = \frac{3}{2} x^{\frac{1}{2}}$,

$[f'(x)]^2 = \frac{9}{4} x$

$L = \int_0^4 \sqrt{1 + \frac{9}{4} x} \, dx = \int_0^4 \frac{\sqrt{4+9x}}{2} \, dx$

$u = 4+9x$, $du = 9\,dx$, $dx = \frac{du}{9}$

when $x=0$, $u=4$; when $x=4$, $u=40$

$$\int_4^{40} \frac{\sqrt{u}}{18}\,du = \frac{1}{18}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_4^{40} = \frac{1}{27}\,u^{\frac{3}{2}}\Bigg|_4^{40}$$

$$= \frac{1}{27}\left((\sqrt{40})^3 - 8\right)$$